What mathematical knowledge could not be

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Introduction

This survey paper will critically discuss four different strategies to explain our knowledge of mathematics. In the first section I will outline *Benacerraf's dilemma* as put forth in Benacerraf's famous paper "Mathematical Truth"[1] – a dilemma faced by any account of mathematical knowledge. The aim of this section is to clarify and discuss the semantic and epistemic constraints that Benacerraf (explicitly and implicitly) imposes, and show how they give rise to his well-known dilemma. In the second section I will review four strategies to overcome this dilemma as they occur in the philosophical literature. The first two platonistic strategies comply with the semantic constraint but, I will argue, provide insufficient answers to the epistemic constraint, while the other two, nominalistic strategies either reject the idea of mathematical knowledge altogether or fail the semantic constraint. In the last section, I will elicit, on the basis of my discussion of the four conceptions, what I label the *fundamental assumption*. I will argue that it is presupposed by all four strategies and suggest that a rejection of this assumption will give rise to a different type of platonistic response. A thorough discussion of this fifth alternative will, however, be postponed to another occasion.

1 Benacerraf's dilemma

In his seminal paper "Mathematical Truth"[2] Benacerraf outlines a dilemma that every account within the philosophy of mathematics faces. The dilemma arises from the need for any such conception to satisfy two constraints whose mutual resolution, however, seems impossible. The first constraint concerns the semantic theory adopted for our mathematical discourse. Here the demand is to have a "homogenous semantical theory in which the semantics for the statements parallel the semantics for the rest of the language." (p.403)

[1] (Benacerraf, 1973)

[2] See (Benacerraf, 1973) all unspecified references in this paper will be to the paper as published in (Benacerraf and Putnam, 1983).

The second constraint is epistemological in nature and demands that "the account of mathematical truth mesh with a reasonable epistemology." (p.403) Benacerraf's dilemma, in its simplest form, arises when one attempts to conform to those two constraints by appeal to the prevalent views in semantics and epistemology: The standard view to comply with the first constraint – a Tarskian theory of truth – imposes an ontology which is incompatible with the standard view in epistemology, which, at the time of Benacerraf's publication[3], comprised causal constraints on knowledge of any type of object. In order to explain the dilemma and the resulting challenge more clearly, I will have a closer look at the two main constraints which each involve two components.

Although not explicitly in his writing, Benacerraf seems to argue for two requirements that jointly make up the semantic constraint.[4] The first requirement is that a theory of truth should be adopted that applies to any discourse, be it an empirical, a mathematical, or even an ethical discourse.[5] This requirement can be found in the following passage:

> "Another way of putting this first requirement is to demand that any theory of mathematical truth be in conformity with a general theory of truth – a theory of truth theories, if you like, – which certifies that the property of sentences that the account calls "truth" is indeed truth. This, it seems to me, can be done *only* on the basis of some *general theory for at least the language as a whole.*" (p. 408, my italics)

The only candidate to satisfy this demand is, according to Benacerraf, a Tarskian account of truth, where truth is spelled out by appeal to reference and satisfaction. Here, the basic idea is to assign semantic values to the different semantic components of a sentence and to then correlate the truth or falsity of a sentence with those semantic values.[6]

The second requirement of the semantic constraint adopted by Benacerraf is to take the surface grammar of sentences of the discourse at face value. Consider his two examples:

1. There are at least three large cities older than New York.

2. There are at least three perfect numbers greater than 17.

If we take the surface grammar at face value we should regard both sentences as having the same structure, namely:

[3] See for example (Goldman, 1967) or (Harman, 1973) as mentioned by Benacerraf.

[4] See especially (Hale and Wright, 2002) who distinguish these two issues. I draw on parts of their discussion of Benacerraf's dilemma.

[5] So a similar "Benacerraf's dilemma" can also be posed for ethical and modal discourse.

[6] Much more can be said about a Tarskian theory of truth, but the details are not important here.

3. There are at least three F/G's that bear R to a.

Benacerraf reinforces this second requirement when he writes that "we should not be satisfied with an account that fails to treat (1) and (2) in parallel fashion, on the model of (3)." (p. 408) Hence, in addition to adopting a Tarskian – referential – theory of truth, the semantic constraint also involves the demand to respect the surface grammar and hence to take it without further qualification that what *seem* to be singular terms, such as "New York" and "17", *are* singular terms.

To sum up, the semantic constraint demands a general and systematic theory of truth to be applied to the mathematical language; and it also demands that we respect the surface-grammar of the mathematical discourse in so applying our theory of truth. Consequently, the *standard view* which satisfies these demands does interpret mathematics by analogy with empirical sentences. Both are analysed using predicates, quantifiers and singular terms and satisfaction-conditions for truth. Crucially, in the case of mathematics, number-terms are regarded as singular terms and as such are required to denote objects. In addition, the objects that number-terms refer to are intuitively regarded as non-spatio-temporally located and are thus abstract. The resulting position – the standard view – is one which genuinely deserves the label 'platonistic', as Benacerraf notes.

The merit of the standard view for the semantic constraint is, according to Benacerraf, that the semantics of mathematics nicely meshes with that of other discourses. Consequently, mixed discourses – where mathematical and empirical terms are used in the same context – pose no additional problem. Further to these virtues of parsimony and simplicity, Benacerraf notes that a Tarskian theory of truth "is the only viable systematic general account we have of truth" in order to account for the first requirement. Finally, note that taking the surface-grammar at face value keeps the truth-conditions of mathematical discourse in line with the thoughts intuitively expressed by such sentences. Hence, respecting the surface-grammar guarantees that what *seems* to be the subject of our mathematical discourse *is* its subject and thus what *seems* to be known and thought about by the subject *is* what is known and thought about by the subject.

Let us now turn to the second *epistemological* constraint proposed by Benacerraf. At first sight, it merely comprises a minimal assumption that hardly seems controversial: we have mathematical knowledge, which "is no less knowledge for being mathematical" (p.409). Subsequently, the demand of the second constraint is just that the account of mathematical truth be compatible with an account of knowledge that renders such truths knowable.

According to Benacerraf, the *standard way* of complying with this second constraint, and the only viable general account of knowledge, is a causal account. The idea, roughly, is that for a subject X to know that p, there has to obtain some type of causal relation between the subject X and the objects, or other items, involved in the subject matter of p. This account can be easily motivated by appeal to our knowledge of medium-sized, everyday objects, which does seem to involve causal relations.

However, adopting the standard view – a Tarskian theory of truth – to comply with the semantic constraint and adopting the standard way – causal theory of knowledge – to comply with the epistemological constraint seems to lead to an impasse, and brings *Benacerraf's dilemma* to the fore: If the truth-conditions of mathematical statements are given by the standard semantical account, then an arithmetical statement involving number terms should be regarded as making reference to numbers as objects that exist in an abstract realm. However, these objects seem unaccountable within the standard view in epistemology, as clearly no causal connection between an object of this sort and the presumably knowing subject can be made out. Thus, no explanation of how we can come to know that the truth-conditions of a mathematical statement obtain can be offered. Crucially, note that what Benacerraf outlines is a genuine dilemma in that *either* of the two constraints (or even both) can be relaxed – either of the two standard views can be dismissed and a new theory of truth or new epistemology can be put into its place. It is not a direct attack (as often thought to be) on the platonistic view of semantics.

Before I outline various general strategies for resolving the dilemma, I will briefly consider a misled interpretation of the epistemological constraint. This will lead to a strengthening of the simple version of the dilemma.

It would be an inadequate platonist response to the dilemma to merely reject the causal conception of knowledge in the light of recent criticisms. Certainly, the causal conception of knowledge does not always seem adequate for medium sized objects which provided its initial motivation, nor is it *prima facie* compatible with knowledge of the future or even facts that can be construed as involving arguably less suspect abstract objects such as the University of St Andrews, Dundee United F.C. or Apple Computers. Thus, so the platonist could argue, the causal theory of knowledge should not be adopted, since it clearly fails to be a simple and general account of knowledge that would qualify as the "only viable systematic general account we have of" knowledge. Consequently, Benacerraf's dilemma would vanish in its initial form since there are no fully general causal constraints on knowledge.

Although I am in general sympathetic to the idea that the causal theory of knowledge is insufficient as an account of our knowledge in general, this response, however, misses the crucial point of Benacerraf's dilemma. The dilemma, in its strongest form, not only points to an incompatibility between two standard views within two areas in the philosophy of mathematics: it also aims to highlight an *integration problem* in the philosophy of mathematics.[7] Accordingly, the epistemological constraint does not necessarily

[7] This terminology was first introduced by (Peacocke, 1999) who himself refers to Be-

depend on adopting a causal account of knowledge. Rather, what is needed is an epistemological account that is able to *integrate* mathematical truths and thereby provide an *explanation* of how it is possible that mathematical truths, whose truth-conditions are spelled out using platonistic ontology (according to the the standard view of the semantic constraint), can be known to obtain by the subject.[8] Rejecting the causal account does little to *explain* how such truth-conditions can be *reliably known* to obtain.

It is this stronger version of the Benacerraf dilemma – in the form of an integration problem – which will be the focus of the following sections. But, for clarity, I believe we should distinguish two requirements towards providing a fully integrated account. An epistemology for mathematics – which involves abstract objects – not only needs to explain how we know that the truth-conditions of a mathematical statement obtain, but, in addition, it also needs to explain how we can have (directed) beliefs about the abstract objects that make the mathematical statements true. To put the point differently, it needs to explain how in thought we can have access or refer to objects that are not spatio-temporal.[9]

This idea can be further explicated by considering the following passage of Hartry Field's interpretation of the dilemma,[10] which also dispenses with the purely causal constraint. He writes:

> "Benacerraf's challenge – or at least, the challenge which his paper suggests to me – is to provide an account of the mechanisms that explain how our beliefs about these remote entities can so well reflect the facts about them." ((Field, 1980), p. 26)[11]

Here, it seems to me, two issues are at hand in the challenge. The first issue is to explain how, in principle, we can successfully talk or have beliefs about abstract objects in the first place. The second issue is to explain, more specifically, how we are justified in thinking that those mathematical beliefs are true, or, as Field would put it, how those beliefs reliably track the mathematical facts.

In conclusion to these considerations, I think we can now see that there are four separate requirements (the first two involved in the semantic con-

nacerraf's dilemma as a prototype of the integration problem.

[8]Note that a solution to the integration problem need not start with a semantic theory as I do here. It is open to start with an epistemology and then aim to integrate a semantic theory within it.

[9]The question of what our mathematical beliefs are about is underlying Benacerraf's earlier discussion in (Benacerraf, 1965).

[10]See (Field, 1980) especially the introduction to this book, p. 20-30

[11]Dialectically there is a difference between Field's presentation and mine. He continues the above quote in the following way: "The idea is that *if it appears in principle impossible to explain this*, then that tends to *undermine* the belief in mathematical entities, *despite* whatever reason we might have for believing in them." (op.cit.) I don't think that this is how Bencacerraf's dilemma should be understood. I believe it is a genuine dilemma while Field regards it more or less as a challenge to the platonist.

straint and the other two in the epistemological constraint) that collectively make up Benacerraf's dilemma and that collectively need to be addressed:

1. *Homogeneous semantic theory*
   The demand that we adopt a general and systematic theory of truth, which – for Benacerraf – should be a Tarskian Theory of truth.

2. *Surface-grammar*
   The demand to respect the surface grammar of mathematical discourse.

3. *Reference and object-directed thought*
   The demand to explain how the objects posited by the semantic theory can, in principle, be in the range of directed thought and talk of the subjects.

4. *Knowledge*
   The 'integration challenge': The demand to reconcile the truths of the subject matter with what can be known by ordinary human thinkers. Crucial here is to provide an explanation of how a subject can have mathematical knowledge and on what basis the subject can claim such knowledge.

2 Strategies to resolve Benacerraf's dilemma

In this section I will review four different strategies to resolve Benacerraf's dilemma. I will begin with a version of platonism which regards mathematical knowledge as a special kind of knowledge that has its own special source and so is distinct in kind from knowledge of other subject matters.

2.1 *Intuitive* platonism

Intuitive platonism adopts the standard view of the semantic constraint, conceiving of the mathematical language as referring to self-subsistent, abstract mathematical entities, and also respects the surface-grammar of the mathematical discourse. Our knowledge of such 'remote' entities is explained by the fact that in the case of mathematics we are concerned with a special type of knowledge, which in the relevant respect is *basic*. The idea here is to break with the demand that a *generally* applicable account of knowledge is needed for every discourse. Instead, the axioms of mathematics and the rules of inference – from which the theorems of mathematics are derived – are regarded as basic in the sense that they cannot be inferred from, and so be known in virtue of, even more fundamental principles. Rather, a subject's *non-inferential* knowledge of the axioms and the rules of inference has its source in the special faculty of *intution*, which, similar to the faculty of perception, provides direct knowledge of the truth of the basic axioms.

The main proponents of intuitive platonism within the philosophy of mathematics have been Kurt Gödel and more recently Charles Parsons[12]. To characterise this type of platonism more precisely, the following often-cited quotation from Gödel highlights the role intuition is supposed to play here:

> "But despite their remoteness from sense-experience, we do have something like a perception of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e. mathematical intuition, than in sense-perception." (Gödel, 1947), p.483-4

This version of platonism, however, faces various difficulties in providing a satisfactory answer to the two epistemic requirements in Benacerraf's dilemma. For one thing, note the transition in this quotation from knowledge of objects ("the objects of set theory") with which mathematical intuition is concerned to knowledge of the truths of axioms ("the axioms force themselves on us") which Gödel aims to underwrite by the faculty of intuition. Leaving aside what underwrites this transition, it remains unclear what the mark of a successful intuitive grasp of such an abstract object is. Just claiming that we can perceive these objects and thereby regard the mathematical axioms as intuitively compelling, obvious, or as somehow "forcing themselves upon us", seems insufficient as a genuine justification of our cognitive beliefs in the truth of the mathematical axioms.

To highlight this further, consider in analogy the scenario in which we postulate a faculty of "perceiving other minds" which provides immediate knowledge of other minds and thereby accounts for the obviousness or intuitive compellingness of certain beliefs about them. No-one (in their right mind) would regard this as a sufficient explanation and justification of our beliefs about other minds.

Also, and connected to this weakness, intuitive platonism has to be able to account for the *fallibility* of this faculty. It would be insufficient to say that Frege's faculty of intuition did let him down when he postulated Basic Law V as an axiom, without explaining why it did and why it does not in other cases where the target statements are consistent.[13]

Various additional concerns could be raised but what seems to be at the heart of most criticisms[14] is that a postulated faculty of intuition fails to

[12]For example in (Parsons, 1979).

[13]Frege in his *Grundgesetze der Arithmetik* (Frege, 1903) put forth Basic Law V and regarded it as a self-evident (logical) truth. Bertrand Russell showed in 1901 in a famous letter to Frege that his axiom is inconsistent.

[14]See for example (Hale and Wright, 2002) for extensive criticism of Parson's and Gödel's

provide a genuine *explanation* of our access to and knowledge of abstract entities, since it is just built into the faculty of intuition – as a brute fact – that it does enable such access and knowledge. Hence, crudely put, this type of platonism seems more like an acknowledgement of the inability to provide an explanation of our knowledge of mathematics than a genuine solution.[15]

In addition, the *intuitive* platonist also runs the risk of dislocating our mathematical knowledge from everyday and, more crucially, scientific knowledge. How on this view is mathematical knowledge embedded and interactive within the scientific corpus of knowledge? It is this version of the well-known *application problem*, which concerns how mathematical knowledge is applicable in empirical science, that proves especially challenging for the *intuitive* platonist. And exactly at this point is where a new type of platonism can be located, one which rejects both the view that mathematics has it's own epistemology, and that it gives rise to a special sort of *a priori* knowledge.

2.2 ***Naturalised*** **platonism**

Naturalised platonism, whose principal author is Quine[16], regards mathematical knowledge as being on a par with scientific knowledge. So, mathematical knowledge is part of our theoretical knowledge, just like knowledge of physics or chemistry, and the objects of mathematics are theoretical objects just like electrons, neutrinos or strings are theoretical objects posited by the physical theories. Therefore, there is neither the need for a special faculty of intuition to explain mathematical knowledge nor does mathematical knowledge enjoy a special status – as a type of a priori knowledge.

But, crucially, then how does the naturalised platonist explain mathematical knowledge, even if it is merely theoretical? After all, this version of platonism also adopts a Tarskian semantic theory to comply with the semantic constraint of Benacerraf's dilemma and, so, regards the objects of mathematics as abstract objects.

This integration challenge has received an answer by the naturalised platonist in the form of the now well-known *Quine-Putnam* indispensability argument.[17] The argument can be presented as follows:

Premise 1 Mathematics is *indispensable* to our scientific theories, in that they can neither be formulated nor practised without mathematical vocabulary and inferences.

views.

[15]There is however a recent resurrection of the idea of a faculty of intuition in current epistemology in the works of BonJour in his (BonJour, 1998) and (Sosa, 2005), which I won't be able to cover here.

[16]See his (Quine, 1986) but there are various others who hold similar positions, such as (Resnik, 1997) and (Shapiro, 1997)

[17]Locus classicus is (Putnam, 1971). For an extensive discussion and recent defence of the indispensability argument, see (Colyvan, 2001)

Premise 2 If mathematics is indispensable to our accepted scientific theories, then if those scientific theories are true then the mathematics involved in scientific theorising is true.

Intermediate Conclusion 1 If scientific theories are true then the mathematics involved in scientific theorising is true.

Premise 3 Scientific theories are true.

Intermediate Conclusion 2 The mathematics involved in scientific theorising is true.

Premise 4 If mathematics is true, then there are the abstract entities to which it purportedly refers, such as numbers, functions, sets.

Conclusion Abstract entities, such as numbers, functions and sets that are appealed to in mathematical theories which are involved in scientific theorising, exist.

It is in virtue of these pragmatic considerations that the *naturalised* platonist aims to incorporate his platonist conception of mathematics within a naturalised epistemology, whereby all knowledge is merely empirical. Obviously crucial here is that mathematics actually *is* indispensable to science in the relevant respect – a claim that has been challenged by Field, and which will be discussed in the next section. But even granting that mathematics is indispensable, two issues remain: Firstly, is a conception for which every statement is empirical and as such "up for revision" stable and, secondly, if it is stable, how exactly do these pragmatic considerations resolve the two epistemic challenges?

The first question has received much discussion in recent years. (Wright, 1986) argues that a Quinean position – which is a form of global empiricism – is intrinsically incoherent. The argument itself is very intricate and I don't propose to discuss it here. The second question, however, is more pertinent to the current discussion and concerns the adequacy of the *naturalised* platonist answers to the epistemological issues about our knowledge of mathematical objects.

Just like the *intuitive* platonist, the *naturalised* platonist does not provide much in terms of an *explanation* of our access to, or knowledge of, abstract objects. The indispensability argument might at best provide an argument *that* we are justified in thinking that there are abstract objects conceived of as theoretical entities. However, note that on this perspective a plausible element of mathematical thinking is lost. As Frege noted, "in arithmetic we are concerned with objects that we come to know not as something alien, from without through the medium of the sense, rather they are directly posited to reason, which, as its nearest kin, it can completely grasp." ((Frege, 1884), §105 my translation) Hence, the "charm" of mathematics as "the reason's proper study" is lost from a naturalised platonist perspective, as knowledge of mathematics and knowledge of the objects of mathematics is merely justified indirectly by its involvement in scientific theories that are true.

Moreover, it is worth noting that knowledge of mathematics is parasitic upon (global) scientific realism (premise three) which is needed to arrive at the conclusion that mathematics is true. And, even granting the soundness of the indispensability argument, only a small part of mathematics will be justified by its application in science and thus the line between applied and pure mathematics becomes of crucial epistemological significance, since only the former, and not the latter (provided the above indispensability argument is all we have), can be justifiably regarded as true.[18]

Thus, one misgiving about naturalised platonism is that, even granting for the moment that the indispensability explains how a subject can have knowledge of parts of mathematics, this position is failing to address Benacerraf's dilemma in full generality. The challenge is how mathematics in general can be integrated into a thinker's corpus of knowledge, and not how some parts can be so integrated. So, I think a position that aims to explain all of mathematics and so tackles Benacerraf's dilemma in full generality is what is needed and desirable. Also, I find it hard to regard the indispensability of mathematics as an adequate explanation of our knowledge of mathematical entities and mathematical statements. Appealing to the need for mathematics in a presumably truth-apt scientific discourse does not – so at least it seems to me – provide the *right type* explanation of a thinker's access to and knowledge of mathematical entities. Lastly and rather worringly, the very idea that mathematics is *indispensable* to science is challengeable (and we shall review this challenge to naturalised platonism in the next section). Therefore, I think that these misgivings raise serious doubts as to whether the naturalised platonist position provides an adequate response to Benacerraf's dilemma.

To summarise, both intuitive and naturalised platonist positions suffer from an inability to provide satisfactory answers to the epistemic issues of Benacerraf's dilemma. Both adopt the standard view to comply with the semantic constraint, yet they fail to account for the epistemological challenges. These failures can be regarded as motivating two alternative positions next to be discussed. The first, *error-theoretic* nominalism, aims to avoid the problematic ontology, without thereby rejecting the standard view in semantics, by claiming that mathematics, taken at face-value, is actually false. This conception is – in some respects – in the tradition of *naturalised* platonism in that it regards all knowledge as theoretical, if it exists at all, but rejects the *indispensability* of mathematics to science, thereby leaving no

[18]There are further issues, for example how and if at all classical logic could be justified in virtue of application in science. See for example (Shapiro, 2005) for such discussion.

possible theoretical reason to accept mathematical entities in the first place. The second conception I shall call *reconstructive* nominalism. It also accepts a Tarskian account, but regards the surface-grammar of the mathematical discourse as misleading. Mathematical ontology, accordingly, is not what it seems to be.

2.3 *Error-theoretic* nominalism

This strategy is famously proposed and explored by Hartry Field in several of his writings.[19] It can be motivated as a reply to the *naturalised* platonist who, according to the nominalist, has not gone far enough. What motivated *naturalised* platonism was the idea that we only have theoretical, i.e. empirical knowledge and thus no additional faculty has to be appealed to in order to account for mathematical knowledge. The *error-theoretic* nominalist adopts the main feature of *naturalised* platonism – that there is only empirical knowledge – while additionally avoiding any commitments to abstract entities whose knowledge is hard to explain.

As the *naturalised* platonist conception was characterised above, the *only* reason it provides for thinking that there are numbers, sets, etc is the previously outlined *indispensability* argument. Field regards this a valid argument but denies its soundness by rejecting the first premise – the claim that mathematics is indispensable to science. Crucially, Field works with an understanding of the underlying notion of indispensability which has to do not merely with the expressive resources gained by using mathematics, but also with the fact that mathematics is *essential* in establishing (proving) theorems and making predictions. Hence, to undermine the indispensability of current mathematical theory, it needs to be shown that there is an alternative theory that does equally well in establishing theorems and making predictions, but which does not involve commitments to numbers, sets, etc.

Field attempts exactly this and provides a framework that, according to him, does equally well, but that makes (arguably) no reference to abstract objects.[20] Consequently, Field rejects the indispensability argument and with it he rejects what the regards as the only good motivation to believe in abstract objects. Hence, he adopts an error-theory for standard mathematics in that, taken at face-value, mathematics is false since it has ontological commitments to things that we have no reason to believe exist.[21] Never-

[19]See his (Field, 1989) and (Field, 1980). Chapter 2 of (Field, 1982) provides a non-technical introduction.

[20]Field is able to provide a theory without reference to mathematical objects that is able to capture Newtonian physics. Field's theory, however, makes reference to space-time points that, some have argued, are best regarded as abstract entities. I won't pursue this line of worry here.

[21]It can be challenged that this consideration suffices to establish an error-theory about mathematics. It seems rather that *agnosticism* should follow from this consideration. I will leave further discussion of this issue aside and grant that an error-theory about

theless, he thinks that we are still entitled to use a false theory and that it is desirable to do so since it is simpler and helps to "speed up inferences" so long as it is *conservative*. The relevant notion of conservativeness is the following:

> *Field's notion of conservativeness*
>
> "A mathematical theory M is conservative iff for any assertion A about the physical world and any body N of such assertions, A doesn't follow from N + M, unless it follows from N alone." ((Field, 1982), p. 58)

The idea is that if mathematics is conservative, it is acceptable to use mathematics since it won't lead to any conclusions that could not be arrived at without mathematics. So, mathematics can be used to "speed-up" inferences or, in general, to make life easier for scientists without the need to endorse its truth and thereby its ontological commitments. Consequently, the basic notion for the *error-theoretic* nominalist is conservativeness and his credo is that "a mathematical theory must be conservative but need not be true".[22]

As can be expected of such a radical view, there is an extensive literature on Field's approach[23] that I won't attempt to survey here. Rather I will assess how such an account would resolve Benacerraf's dilemma, assuming the more specific criticisms are resolved.

In some respects, Field's resolution of the dilemma is simple but radical: We are misled in thinking that mathematics is true – quite the opposite, it is false. Still it is valuable because it is conservative (or at least parts of it are). Just like the two types of platonism above, the *error-theoretic* nominalist adopts the standard view to address the *semantic constraint*. He adopts a Tarskian theory of truth and he respects the surface grammar – but denies the truth of the mathematical discourse, while granting its usefulness cashed out in terms of *conservativeness*. So, provided *error-theoretic* nominalism can overcome various technical difficulties in order to account for enough science, consistent with maintaining that mathematics is false but conservative, this position aims to overcome Benacerraf's dilemma by a radical route: denying mathematical knowledge altogether. Moreover, there is no need to account for a thinker's reference to and object-directed thought about mathematical entities.

mathematics can be motivated in this way.

[22](Field, 1982), p. 58.

[23]A chronological collection of the most important literature is: (Malamet, 1982), (Shapiro, 1983), (Shapiro, 1984), (Hale, 1987), as well as most of the papers in the collection (Irvine, 1990). For a very detailed survey of Field and his critics consult (MacBride, 1999) and for a detailed account and criticism of the technical framework see (Urquhart, 1990), as well as (Burgess and Rosen, 1997) who also provide a very nice reconstruction of the nominalism-platonism debate. For some replies (especially to Shapiro) see (Field, 1985)

However, is this a promising strategy to resolve Benacerraf's dilemma? I don't think so. A satisfying solution to the dilemma should not consist in giving up the basic assumption that we have mathematical knowledge. In a similar vein to my criticism to *naturalised platonism*, I think what is needed is a direct solution – taking Benacerraf's challenge head on – and integrate mathematical knowledge in general. Error-theoretic nominalism, in contrast, merely acknowledges defeat by dropping Benacerraf's main assumption that mathematical knowledge "is no less knowledge for being mathematical" (p.409). In turn, it seeks to explain why, in spite of there being no mathematical knowledge, we can pursue mathematics without a bad conscience. So, I think, in the context of a resolution to Benacerraf's dilemma, Field's approach is a mere last resort. I will therefore, leave aside *error-theoretic* nominalism and continue to explore the possibility for a more suitable resolution of Benacerraf's dilemma. *Reconstructive* nominalism is an alternative version of nominalism that shares the general scruples about abstract objects, but, less radically than Field, aims to retain the truth of mathematics, even if it is not truth in virtue of the properties of numbers, sets, functions, etc. This strategy will be discussed in the next section.

2.4 ***Reconstructive*** **nominalism**

The *reconstructive* nominalist view can also be regarded as a response to *naturalised* platonism. Yet, it adopts a different strategy to avoid the problematic ontology, which seems to pose a serious problem for our knowledge of mathematics and so for a satisfying solution to Benacerraf's dilemma. In contrast to *error-theoretic* nominalism that rejects the problematic ontology and the *indispensability* of mathematics, the *reconstructive* nominalist trades ontology for ideology. To explain, he rejects the following crucial move in the *indispensability* argument:

Intermediate Conclusion 2 The mathematics involved in scientific theorising is true.

Premise 4 If Mathematics is true, then there are the abstract entities to which it purportedly refers, such as numbers, functions, sets.

Conclusion Abstract entities, such as numbers, functions and sets that are appealed to in mathematical theories which are involved in scientific theorising, exist.

The *reconstructive* nominalist challenges premise four that the truth of mathematics commits one to the objects purported to be referred to by the mathematical terms. And to underwrite the motivation for rejecting this conditional, the *reconstructive* nominalist discards the second requirement of the semantic constraint, which involves the demand to take the surface-grammar of the mathematical discourse at face-value.

Rejecting this assumption opens up the possibility of *reconstructing* mathematical discourse in various ways. One proponent of this strategy is Hellmann who adopts a version of *Modal Structuralism*[24]. The idea is to trade mathematical ontology – reference to numbers, sets, etc – for added ideology, namely the use of modal discourse. Mathematics is now conceived as concering *possible* structures (and objects).

This conception is indeed a type of nominalism, as it refrains from reference to and quantification over existing abstract objects and instead merely commits one to possible entities. In the case of arithmetic, the commitment is to a possible ω-structure – a structure that exhibits the properties normally assigned to numbers and so makes the Peano axioms true.[25] Various formal details need to be attended to, to make this reconstructive strategy work.[26] Here, however, I will leave these formal issues aside and, just as above, note various difficulties with this approach and assess how it fares with regard to overcoming Benacerraf's dilemma.

Reconstructive nominalism typically adopts Tarski's theory of truth but denies the need to respect the surface grammar. Hence, such a conception denies the basic presumption that what seems to be referred to, or what seems to be thought about when doing mathematics, *is* what is referred to or thought about. According to reconstructive nominalism, there are no such objects as numbers underlying our mathematical thought and talk. We are systematically misguided in thinking that the surface grammar represents reality. To some, this might seem like a hard bullet to bite and, in general, I believe that an account which does respect the surface-grammar has advantages over a reconstructive account.

The important issue, however, is the epistemic constraint. The basic idea is that by avoiding the critical ontology (abstract objects) by appeal to possible structures, this type of *reconstructive* nominalism can overcome Benacerraf's dilemma. So, the thought had better be that knowledge of merely *possible* structures is easier explained and justified than knowledge of actual abstract objects. I think, however, that exactly this claim can be challenged.[27] First, to clarify the modal structuralist position, it should be

[24](Hellman, 1989)

[25]On one reading of nominalism, namely Goodman's and the early Quine's, this approach would not be considered nominalistically acceptable. Compare: "Goodman and I got what we could get in the way of mathematics on the basis of a nominalist ontology and without assuming an infinite universe. We could not get enough to satisfy us. But we would not for a moment have considered enlisting the aid of modalities. The cure would have been far worse than the disease." (reply to Charles Parson in (Hahn, 1986) as quoted from (Burgess and Rosen, 1997), p. 248. I won't here enter the dispute what is distinctive of nominalism, and continue to regard Hellman as a nominalist.

[26]For example providing the right type of translation from standard mathematics to modal statements, the problem of trivial conditionals, etc. For details see (Hellman, 1989) and (Burgess and Rosen, 1997).

[27]See especially (Hale, 1996).

noted that the possible structures are structures of objects. For there to possibly be an ω-structure to make the Peano axioms true, there have to be possibly infinitely many objects. Otherwise structures *per se* seem every bit as abstract as numbers, sets, etc. As a nominalist, Hellman should not be committed to the possible existence of infinitely many *abstract* objects because it would follow that it is contingent whether there are abstract objects – a curious contingency that needs to be explained.[28] However, Hellman is aware of these difficulties and commits himself to the possibility of a *concrete* ω-structure, i.e. that there could be infinitely many concrete objects (making up an ω-structure).

But, then how can one know this modal claim? What explains our knowledge, if we have it, that there could be infinitely many concrete objects? One option is to argue from the mere conceivability of there being infinitely many concrete objects that it is possible that there are – but the extent to which conceivability is a good guide to possibility is a further issue. In general we can say that, just as in the case of abstract objects, a variant of Benacerraf's dilemma can be raised for the modal realm and the possible existence of certain objects.[29] Unless it is clear that the latter type of knowledge – modal knowledge – is easier to explain and that the general scruples about abstract objects should be upheld, there are no advantages (but merely disadvantages based on added complexity and the rejection of surface-grammar) to such a reconstructive approach.[30]

2.5 The fundamental assumption of the four strategies

We have seen that apart from intuitive platonism, the other three strategies either truncate the knowledgeable part of mathematics, deny any mathematical knowledge, or turn mathematical knowledge into modal knowledge. Hence, these three strategies have not taken the dilemma head on; rather, they give up on basic components that gives rise to the dilemma. Considering that neither these indirect responses nor the one direct response (intuitive platonism) offer a satisfying account to resolve Benacerraf epistemic constraints, I here want to explore whether there is an assumption that is shared by these four strategies. By identifying and then challenging it, we might be able to arrive at an alternative conception that holds the key to resolving Benacerraf's challenge. I think, that each strategy is committed to the following conditional, which I will label the *fundamental assumption*:

[28]This is a point made in Hale (Hale, 1996). Consult this paper for further discussion and explanation.

[29]See for an illuminating discussion (Stalnaker, 1996).

[30]I have have to forgo discussion of other structuralist views, such as those of (Shapiro, 1997) and (Resnik, 1997), and the more pessimistic position which claims that there is no, or not just one solution, to the Benacerraf challenge, defended in (Azzouni, 1994) and (Balaguer, 1998) respectively.

> If there is a priori mathematical knowledge and the mathematical discourse is construed at face-value, then there has to be some form of acquaintance with the objects involved that underwrites this knowledge.

This assumption can reasonably explain why the intuitive platonist postulates a "perception-like" faculty – the faculty of intuition – which provides a form of acquaintance with the abstract objects. Having engaged in this (mysterious) interaction with the abstract objects results, according to intuitive platonism, in a priori knowledge of the abstract objects and the relevant axioms purported to be about these objects. Naturalised platonism also accepts this conditional but, in contrast to intuitive platonism, denies that there can be a perception-like mathematical faculty or any other form of acquaintance with abstract objects, and accordingly denies that there can be any type of a priori knowledge of mathematics. As a result, there is a need to resort to broadly empirical and scientific considerations to explain mathematical knowledge. Both error-theoretic and reconstructive nominalism challenge the sufficiency of a broadly empirical epistemology of mathematics and both nominalist positions can also be interpreted as adopting the fundamental assumption. Error-theoretic nominalism adopts an even stronger assumption, namely that any type of knowledge has to involve some form of interaction or acquaintance, however indirect. Since there can be no such interaction with mathematical entities there is no mathematical knowledge in general. Reconstructive nominalism in contrast also accepts the truth of the fundamental assumption but then denies the component of the antecedent which claims that mathematics is to be construed at face-value.

Reflecting on this assumption offers the opportunity to rethink the platonist strategy. What exactly would be involved in rejecting the truth of the fundamental assumption? If it were rejected, it should be possible to have a priori knowledge of mathematics and the mathematical objects without having an acquaintance with the objects that mathematics is about. So, there is no need to account for an initial interaction with the subject-matter in order to found knowledge of that subject-matter.

Thus a challenge of this *subject-matter first* idea would comprise the claim that a subject can have and justifiably claim knowledge of numbers without direct acquaintance as *intuitive* platonism demands and without resorting to purely pragmatic considerations which threaten a genuine explanation of mathematical knowledge. Instead, so the suggestion might go, we can have knowledge of numbers by reflecting upon statements about the objects in question – that is we can, in some sense, gain knowledge by linguistic competence and by mastering of the mathematical discourse. I think pursuing this new (fifth) strategy is definitely worth a try[31] and some philosophers have attempted to take this *linguistic turn* for our knowledge

[31] After all, it resulted in a PhD for the author of this paper.

of mathematics, yet a proper discussion of this strategy will have to be postponed to another occasion.[32]

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[32] Frege in his *Grundlagen der Arithmetik* ((Frege, 1884)) has often been identified as the first proponent of this fifth strategy. The so-called Neo-Fregeans are defending a view that is broadly in line with the fifth strategy. See especially (Wright, 1983), (Hale, 1987) and (Hale and Wright, 2001). It has been discussed and criticised extensively in recent literature, especially by Dummett (see his (Dummett, 1981) and (Dummett, 1991)). An excellent survey article about Neo-Fregeanism can be found in (MacBride, 2003). An up-to-date bibiography discussing many different issues of Neo-Fregeanism can be found on the Arche website: http://arche-wiki.st-and.ac.uk/ ahwiki/bin/view/Arche/MathsBibliographies

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